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The behaviour of similar solutions in a compressible boundary layer

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This paper discusses the mathematical properties of similar solutions of the boundary-layer equations in a compressible model fluid, under assumptions first introduced by Stewartson and by Li & Nagamatsu. Assuming a favourable pressure gradient and that backflow is not present, our results include (among other things) a rigorous proof that velocity overshoot occurs in the boundary layer if the wall is heated, and that this is true whether or not suction, blowing or slipping occurs at the wall; while, conversely, velocity overshoot does not occur when the wall is cooled and the amount of slipping at the wall is suitably restricted.

1. Introduction

We shall be concerned in this note with the boundary layer of a compressible fluid adjacent to a fixed wall. Under appropriate conditions on the physical properties of the fluid one can look for similar solutions to this problem. As was first shown by Stewartson (1949) and by Li & Nagamatsu (1955) this leads to the linked pair of non-linear differential equations

$$f''' + ff'' + \beta(h - f'^2) = 0, \qquad (1)$$

$$h'' + fh' = 0, \qquad (2)$$

with the boundary conditions

$$f(0) = f'(0) = 0, \quad h(0) = a; \quad f'(\infty) = h(\infty) = 1. \qquad (3)$$

Here f' and h are effectively the tangential velocity and the total energy function in the boundary layer, β is a constant dependent on the particular stream conditions, and a measures the ratio of the temperature at the wall to the stagnation temperature in the mainstream. The first two conditions of (3) express the fact that the fluid adheres to the wall without slipping. (It is also possible to consider walls which are permeable to the fluid and even to allow a certain amount of slipping. The changes in the argument which are required to handle these more general cases are discussed briefly at the end of the paper.)

A derivation of the above equations and a discussion of the stream conditions under which they are valid can be found in several standard works, e.g. Schlichting (1960), Rosenhead (1963) and Stewartson (1964). We remark here only that the constant β is non-negative when the pressure gradient in the mainstream is favourable.

If $\beta = 0$, equation (1) becomes Blasius's equation, the existence, uniqueness, and behaviour of the solution being well known. Solutions of (1), (2), (3) have been calculated numerically for various values of the parameters a and β by Li & Nagamatsu (1955) and by Cohen & Reshotko (1956). In addition, Ho & Wilson (1967) have recently proved the existence of solutions when $\beta > 0$ and $0 < a \leqslant 1$, that is, when the wall is cooled, while in work to appear shortly the present authors have extended this result to the case when the wall is heated ($a > 1$) and slipping, suction and blowing are present.

For the case of a heated wall with $\beta > 0$ numerical work and physical reasoning indicate the existence of velocity overshoot in the boundary layer. (By velocity overshoot we mean the occurence of $f' > 1$ for some range of values of the independent variable η.) An asymptotic analysis of this phenomenon was given by Cohen & Reshotko (1956). While this analysis is valuable, and suggestive of the general behaviour to be expected of solutions, it nevertheless was based on dropping certain terms from the equations and accordingly is not entirely convincing. This being the case, one of the main purposes of this note is to show that such an overshoot must in fact occur as a direct and mathematically exact consequence of the differential equations and the fact that $a > 1$. At the same time we shall consider the general qualitative behaviour of solutions when $\beta > 0$, both when the wall is heated and when it is cooled. It will be assumed throughout that the solutions under discussion do not exhibit backflow, that is, we assume that $f' \geqslant 0$ for all solutions under consideration.

When $0 < a < 1$ we shall show, specifically, that the functions h and f' are monotonically increasing and concave, with $f'^2 < h$. It follows, in consequence, that similar solutions do not exhibit velocity overshoot when the wall is cooled. (When $a = 1$, see the remark at the end of §3.)

When $a > 1$ we show that h is monotonically decreasing and convex. Moreover, f' first increases to a maximum value greater than 1 and then decreases monotonically to its limit value 1. In addition $f'^2 < h$ everywhere, and the graph of f' is concave in its rising portion, has exactly one inflexion when the parameters β and a satisfy the relation

$$2\beta \leqslant \frac{\sqrt{a}}{\sqrt{a}-1},$$

and otherwise has at most a finite (odd) number of inflexions.

The condition $f'^2 < h$, which occurs both for cooled and for heated walls, has the physical interpretation that the local Mach number in the boundary layer is less than the Mach number in the mainstream. To see this, we recall the relationship of f' and h to the physical variables in the flow, namely

$$u = Uf', \quad \tfrac{1}{2}u^2 + c_p T = Hh,$$

where u is the tangential velocity and T the temperature, c_p the specific heat at constant pressure, and H and U mainstream scaling parameters. A simple calculation then shows that $f'^2 < h$ is equivalent to

$$u^2/T < U^2/T_\infty,$$

which establishes the required interpretation (since in the fluid the squared speed of sound is proportional to temperature).

We emphasize finally that what is being proved here is that *if solutions exist which do not exhibit backflow* then they must behave in the manner described. The existence of solutions is another matter, dealt with in the papers noted earlier, while the uniqueness of solutions remains an open question.

2. Existence of velocity overshoot

We start by considering the existence of velocity overshoot when $a > 1$. From (2) there follows

$$h'(\eta) = -C\exp\left(-\int_0^\eta f(v)\,dv\right), \tag{4}$$

where the real coefficient C must be positive since $h(0) = a > 1$ and $h(\infty) = 1$. Since $f' \to 1$ as $\eta \to \infty$ we have $f(\eta) \sim \eta$, whence by L'Hôpital's rule

$$\{h-1\}.\left\{\eta\exp\left(\int_0^\eta f\,dv\right)\right\} \to C \quad \text{as } \eta \to \infty. \tag{5}$$

Now set

$$F(\eta) = f''(\eta)\exp\left(\int_0^\eta f(v)\,dv\right)$$

and write (1) in the form

$$F' = -\beta(h-f'^2)\exp\left(\int_0^\eta f(v)\,dv\right). \tag{6}$$

If we suppose, contrary to what has to be proved, that overshoot does not take place, then $f' \leqslant 1$ everywhere. Since $f' \geqslant 0$ by assumption, we find $f'^2 \leqslant 1$ and so, by (5) and (6),

$$F' \leqslant -\beta(h-1)\exp\left(\int_0^\eta f(v)\,dv\right) \leqslant -\beta C/2\eta$$

for η sufficiently large. Integrating this yields $F \to -\infty$ as $\eta \to \infty$, whence f'' is ultimately negative. This in turn implies (using the condition $f' \to 1$ as $\eta \to \infty$) that $f' > 1$ for all sufficiently large η, and we have the required contradiction.

3. The inequality $f'^2 < h$

We show now that $f'^2 < h$ when $\beta > 0$, irrespective of whether the wall is heated or cooled. Letting $G = h - f'^2$, an easy calculation from (1) and (2) gives

$$G'' + fG' - 2\beta f'G = -2(f'')^2. \tag{7}$$

Suppose for contradiction that $G \leqslant 0$ at some point. Since $G = a > 0$ at $\eta = 0$ and $G = 0$ at $\eta = \infty$, there must then be some point η_0 at which G takes a negative or zero minimum. At this point

$$G \leqslant 0, \quad G' = 0, \quad G'' \geqslant 0,$$

contradicting (7) unless $f''(\eta_0) = 0$. But then $G' = h' - 2f'f'' = h'$ at η_0, contradicting the fact that $G' = 0$ at η_0.

4. The cooled wall

Consider now the behaviour of f' and h when $0 < a < 1$. Relation (4) continues to hold, though now the coefficient C must be negative since $h(0) < 1$. Therefore $h' > 0$, and in turn $h'' < 0$ by (2). Thus the graph of h is monotonically increasing and concave.

By (6) the function F is monotonically decreasing. If its limit as η tends to infinity is negative, then f'' must eventually be negative. But (using the condition $f' \to 1$ as $\eta \to \infty$) this contradicts the fact that $f'^2 < h < 1$. Hence the limit value of F is non-negative and F and f'' are positive functions. Finally by (1)

$$f''' = -ff'' - \beta(h - f'^2) < 0.$$

Thus the graph of f' is also monotonically increasing and concave.

Remark

When $a = 1$ we are easily led as above to $h' \equiv 0$ and $h \equiv 1$. Equation (1) then reduces to the Falkner–Skan equation; it follows finally by the argument in the preceding paragraph that f' is monotonically increasing and concave.

5. The heated wall

We turn lastly to the behaviour of f' and h when $a > 1$. From (4) and (2) it is clear that the graph of h is monotonically decreasing and convex.

Also by (6) the function F is monotonically decreasing. Let its limit value be denoted by l (possibly $-\infty$). The case $l \geqslant 0$ is impossible, since then we always have $f'' \geqslant 0$ and $f' \leqslant 1$, which has already been excluded. Hence $l < 0$ and f'' is first positive and then negative (the case where f'' takes only non-positive values obviously cannot occur). Consequently f' must first increase to a maximum value greater than 1 and then decrease to its limit value 1 at infinity.

Since $f''' = -ff'' - \beta(h - f'^2)$ it follows that $f''' < 0$ so long as the graph of f' is rising.

It remains to consider the inflexions in the graph. We show first that these are finite in number and that eventually $f''' > 0$. Indeed, once f' becomes greater than 1 we have

$$f''' \geqslant -ff'' - \beta(h-1).$$

For sufficiently large values of η, however,

$$-ff'' \exp\left(\int_0^\eta f(v)\,dv\right) = -fF \geqslant -\tfrac{1}{2}l\eta \qquad (l < 0)$$

and

$$(h-1)\exp\left(\int_0^\eta f(v)\,dv\right) \leqslant 2C/\eta$$

by the results of §2. Thus f''' is ultimately positive. If there were an infinite number of inflexions they would have a finite limit point. But then (since f is an analytic function) it would follow that $f''' \equiv 0$, which is impossible.

Before proving the final result, we show that

$$2F > -C. \tag{8}$$

To begin with, by the definition of F and G and by (4)

$$G' = h' - 2f'f'' = -(C + 2f'F)\exp\left(-\int_0^\eta f(v)\,dv\right).$$

If (8) were not true, then ultimately G' would be positive. Since $G = 0$ at $\eta = \infty$ this would imply that G is ultimately negative, a situation which has already been excluded.

With (8) established, we now observe that

$$f^{\text{iv}} = -ff''' - f'f'' - \beta(h' - 2f'f'').$$

Let η_1 be an inflexion of f'. By (4) and the definition of F, we have at η_1

$$f^{\text{iv}} = \{(2\beta - 1)f'F + \beta C\}\exp\left(-\int_0^{\eta_1} f(v)\,dv\right). \tag{9}$$

We assert that if

$$2\beta \leqslant \frac{\sqrt{a}}{\sqrt{a}-1} \tag{10}$$

then $f^{\text{iv}} > 0$ at η_1. Consider separately the two cases

$$0 < 2\beta \leqslant 1, \quad 1 < 2\beta \leqslant \frac{\sqrt{a}}{\sqrt{a}-1}.$$

For the first one, note that $f'' = -\beta G/f < 0$ at η_1 according to (1). Hence $F < 0$ at η_1 and the assertion follows at once from (9). In the second case, using (9), (8) and the fact that $f' < \sqrt{h} < \sqrt{a}$, we find

$$f^{\text{iv}} > \tfrac{1}{2}C\{2\beta - (2\beta - 1)\sqrt{a}\}\exp\left(-\int_0^{\eta_1} f(v)\,dv\right) \geqslant 0.$$

Thus when (10) holds, f''' can only increase as we pass an inflexion point. Consequently there can be no more than one inflexion in the graph of f' under this condition. On the other hand, there must be at least one such point in view of the already established behaviour of f'. This completes the proof of the assertions made in the introduction.

6. Generalizations

In the preceding sections we restricted ourselves specifically to the situation when $f(0) = f'(0) = 0$, but the analysis easily extends to the more general case $f(0) = \alpha_1, f'(0) = \alpha_2$, where α_1 is any real constant and α_2 is a non-negative constant satisfying $\alpha_2 \leqslant \sqrt{a}$. (The last condition expresses the fact that the Mach number at the wall should not exceed the Mach number in the mainstream; cf. the remark at the close of the introduction.)

Nothing in §2 is altered by the above changes, and it therefore follows that velocity overshoot continues to hold when $a > 1$. Furthermore, with the help of the condition $\alpha_2 \leqslant \sqrt{a}$ it is not hard to see that the argument of §3 continues to hold; we can therefore conclude that $f'^2 < h$ for all positive values of η.

When $\alpha_1 > 0$, that is, if fluid is sucked from the boundary layer into the wall, nothing needs to be changed in §§4 and 5. The qualitative behaviour of the graphs of f' and h therefore remains unaltered for this case.

On the other hand, if $\alpha_1 < 0$, that is, if fluid is blown through the wall into the boundary layer, then some significant changes result in the graphs of f' and h. In particular, when $\alpha_1 < 0$ the function f will first be negative and then positive, assuming as always that the motion exhibits no backflow. Then, since h' is of one sign, it is apparent from (2) that the graph of h will have exactly one inflexion for $a \neq 1$, being ultimately concave when $a < 1$, and ultimately convex when $a > 1$. Also, following the arguments of §§4 and 5, it is clear that f' will be monotonically increasing when $a \leqslant 1$, while when $a > 1$ it will first increase to a maximum value greater than 1 and then decrease afterwards.

The inflexions of f' are slightly harder to discuss in the present case. We first observe that according to the argument of §5 they must be finite in number whatever the value of a, and that f' will be ultimately concave when $a \leqslant 1$ and ultimately convex when $a > 1$.

When $a \leqslant 1$ and $0 < 2\beta \leqslant 1$ one can show as, in the final part of §5, that f' has at most one inflexion. Similarly, when $a > 1$ and $0 < 2\beta \leqslant 1$, the graph of f' has exactly one inflexion on its falling portion (though it may have other inflexions on its rising part), and when $1 \leqslant 2\beta \leqslant \sqrt{a}/(\sqrt{a}-1)$ it has exactly one inflexion on its entire course. There is no need to elaborate the details of the argument.

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